THE

TWO BOOKS

OF

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APOLLONIUS PERGÆUS,

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CONCERNING

TANGENCIES,

As they have been Reſtored by

FRANCISCUS VIETA and MARINUS GHETALDUS.

WITH A

SUPPLEMENT.

By JOHN LAWSON, B.D.

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BRITISH MUSEUM

PREFACE.

OF the twelve Analytical Treatiſes recited by Pappus in his Preface to the 7th Book of his Mathematical Collections, we have very little of the Originals remaining, viz. only Euclid's Data, and part of Apollonius's Conics. The loſs of the reſt is very much to be lamented by all Lovers of the Mathematics. “Valde quidem dolendum eſt quod reliqui tractatus Veterum Analytici, a Pappo memorati, aut perierint, aut nondum lucem conſpexerint. Nam minime dubito quin eorum nonnulli, Arabicè ſaltem verſi, alicubi terrarum lateant, pulvere magis quam tenebris ſuis involuti.” Dr. Halley's Preface to his Apollonius de Sectione Rationis et Spatii.

Some ingenious men have attempted, from the account of them given by Pappus, to reſtore ſome of theſe loſt Treatiſes. Snellius has endeavoured to give us the Books De Sectione Rationis, De Sectione Spatii, and De Sectione Determinata. Fermat and Schooten have laboured in the Treatiſe De Locis Planis; and Marinus Ghetaldus in that De Inclinationibus. But thoſe who have ſucceeded beſt, and done the moſt this way, are two incom-

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parable Mathematicians of our own Country, Dr. Halley and Dr. Simſon, to whom the World is very much obliged for their Geometrical Labours. The firſt of theſe, from an Arabic MS in the Bodleian Library, has reſtored the Books De Sectione Rationis; and from his own Sagacity ſupplied thoſe De Sectione Spatii: and the other has with equal pains and ingenuity completed thoſe De Locis Planis.

As to the Treatiſe De Tactionibus, which I now give the Engliſh Reader, it has been reſtored by Vieta under the Title of Apollonius Gallus, and his Deficiencies ſupplied by Marinus Ghetaldus. I have endeavoured to do Juſtice to my Authors by all poſſible Care both in the Text and in the Figures; and have added a few Propoſitions of my own, by way of Supplement, in which I have propoſed Ghetaldus's Problems over again without a Determination, and have found the Locus of the center of the circle required, which I have not ſeen done before in any Author.

EXTRACT from PAPPUS's Preface to his Seventh Book in Dr. HALLEY's Tranſlation.

DE TACTIONIBUS II.

HIS ordine ſubnexi ſunt libri duo DE TACTIONIBUS, in quibus plures ineſſe propoſitiones videntur; ſed & ex his unam etiam faciemus, ad hunc modum ſe habentem. "E "punctis rectis & circulis, quibuſcunque tribus poſitione "datis, circulum ducere per ſingula data puncta, qui, ſi fieri "poſſit, contingat etiam datas lineas." Ex hac autem ob multitudinem in Hypotheſibus datorum, tam ſimilium quam diſſimilium GENERUM, fiunt neceſſario decem propoſitiones diverſæ; quia ex tribus diſſimilibus generibus fiunt diverſæ triades inordinatæ numero decem. Data etenim eſſe poſſunt vel tria puncta; vel tres rectæ; vel duo puncta & recta; vel duæ rectæ & punctum; vel duo puncta & circulus; vel duo circuli & punctum; vel duo circuli & recta; vel punctum, recta & circulus; vel duæ rectæ & circulus; vel tres circuli. Horum duo quidem prima problemata oſtenduntur in libro quarto primorum Elementorum. Nam per tria data puncta, quæ non ſint in linea recta, circulum ducere, idem eſt ac circa datum triangulum circumſcribere. Problema autem in tribus datis rectis non parallelis, ſed inter ſe occurrentibus, idem eſt ac dato triangulo circulum inſcribere. Caſus vero duarum rectarum parallelarum cum tertiâ occurrente, quaſi

pars

pars eſſet ſecundæ ſubdiviſionis, cæteris permittitur. Deinde proxima ſex problemata continentur in primo libro. Reliqua duo, nempe de duabus rectis datis & circulo, & de tribus datis circulis, ſola habentur in ſecundo libro; ob multas diverſaſque poſitiones circulorum & rectarum inter ſe, quibus fit ut etiam plurium determinationum opus ſit. Prædictis his Tactionibus congener eſt ordo problematum, quæ ab editoribus omiſſa fuerant. Nonnulli autem priori horum librorum illa prefixerunt: Compendioſus enim & introductorius erat tractatus ille, & ad plenam de Tactionibus doctrinam abſolvendam maxime idoneus. Hæc omnia rurſus una propoſitio complectitur, quæ quidem quoad Hypotheſim magis quam præcedentia contracta eſt, ſuperaddita autem eſt conditio ad conſtructionem: eſtque hujuſmodi. " E punctis, rectis, vel " circulis, datis duobus quibuſcunque, deſcribere circulum " magnitudine datum, qui tranſeat per punctum vel puncta " data, ac, ſi fieri poſſit, contingat etiam lineas datas." Continet autem hæc propoſitio ſex problemata: ex tribus enim quibuſcunque diverſis generibus fiunt Duades inordinatæ diverſæ numero ſex. Vel enim datis duobus punctis, vel duabus rectis, vel duobus circulis, vel puncto & rectâ, vel puncto & circulo, vel rectâ & circulo, opportet circulum magnitudine datum deſcribere, QUI DATA CONTINGAT; hæc autem reſolvenda ſunt & componenda ut & determinanda juxta Caſus. Liber primus TACTIONUM problemata habet ſeptem; ſecundus vero quatuor. Lemmata autem ad utrumque librum ſunt XXI; Theoremata LX.

SYNOPSIS of the PROBLEMS.

Prob. 1. ● ●	Prob. 2. | |	Prob. 3. ○ ○
4. ● |	6. | ○	
5. ● ○		
Euclid ● ● ●	Euclid | | |	Prob. 14. ○ ○ ○
Prob. 7. ● ● |	Prob. 9. | | ○	
12. ● ● ○	11. | ○ ○	
8. ● | |		
13. ● ○ ○		
10. ● | ○		

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Prob. 7. • • |	Prob. 9. | | ○	
12. • • ○	11. | ○ ○	
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13. • ○ ○		
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PROBLEMS

CONCERNING

TANGENCIES.

PROBLEM I.

THROUGH two given points A and B to deſcribe a circle whoſe radius ſhall be equal to a given line Z.

Limitation. 2 Z muſt not be leſs than the diſtance of the points A and B.

Construction. With the centers A and B, and diſtance Z, deſcribe two arcs cutting or touching one another in the point E, (which they will neceſſarily do by the Limitation) and E will be the center of the circle required.

PROBLEM II.

Having two right lines AB CD given in poſition, it is required to draw a circle, whoſe Radius ſhall be equal to the given line Z, which ſhall alſo touch both the given lines.

Case 1ſt. Suppoſe AB and CD to be parallel.

Limitation. 2Z muſt be equal to the diſtance of the parallels, and the conſtruction is evident.

Case 2d. Suppoſe AB and CD to be inclined to each other, let them be produced till they meet in E, and let the angle BED be biſected by EH, and through E draw EF perpendicular to ED, and equal to the given line Z; through F draw FG parallel to EH, meeting ED in G, and through G draw GH parallel to EF. I ſay that the circle deſcribed with H center, and HG radius, touches the two given lines: it touches CD, becauſe EFGH is a Parallelogram, and hence the angle FEG is equal to EGH, but FEG is a right one by Conſtruction. Let now HI be drawn from H perpendicular to AB: then the two triangles EHI and EHG having two angles in one HEI and EIH reſpectively equal to two angles in the other HEG and EGH, and alſo the ſide EH common, by Euc. I. 26. HI will be equal to HG, and therefore the circle will touch alſo the other line AB: and HG or HI equals the given line Z, becauſe EF was made equal to Z, and HG and EF are oppoſite ſides of a parallelogram.

PROBLEM III.

Having two circles given whoſe centers are A and B, it is required to draw another, whoſe Radius ſhall be equal to a given line Z, which ſhall alſo touch the two given ones.

This Problem has various Caſes, according to the various poſition of the given circles, and the various manner of deſcribing the circle required: but there are ſix principal ones, and to the conditions of theſe all the reſt are ſubject.

Case 1ſt. Let the circle to be deſcribed be required to be touched *outwardly* by the given circles.

Limitation. Then it is neceſſary that 2Z, or the given Diameter, ſhould not be leſs than the ſegment of the line joining the centers of the given circles which is intercepted between their *convex* circumferences, viz. not leſs than CD in the Figure belonging to Caſe 1ſt.

Case 2d. Let the circle to be deſcribed be required to be touched *inwardly* by the given circles.

Limitation. Then it's Diameter muſt not be given leſs than the right line, which drawn through the centers of the given circles, is contained between their *concave* circumferences; viz. not leſs than CD.

Case 3d. Let the circle to be deſcribed be required to be touched *outwardly* by one of the given circles, and *inwardly* by the other.

Limitation. Then it's Diameter muſt not be given leſs than the ſegment of the right line, joining the centers of the given circles, which is intercepted between the *convex* circumference of one and the *concave* circumference of the other; viz. not leſs than CD.

Case 4th. Let one of the given circles include the other, and let it be required that the circle to be deſcribed be touched *outwardly* by them both.

Limitation. Then it's Diameter muſt not be given greater than the greater ſegment of the right line, joining the centers of the given circles, which is intercepted between the *concave* circumference of one and the *convex* circumference of the other; nor leſs than the leſſer ſegment; viz. not greater than CD, nor leſs than MN.

Case 5th. Let one of the given circles include the other, and let be required that the circle to be deſcribed be touched *outwardly* by one of the given circles, and *inwardly* by the other.

Limitation. Then it's Diameter muſt not be given greater than the greater ſegment of the right line, joining the centers of the given circles, which is intercepted between the two *concave* circumferences of the ſaid circles, nor leſs than the leſſer ſegment; viz. not greater than CD, nor leſs than MN.

Case 6th. Let the two given circles cut each other, and let it be required that the circle to be deſcribed, and to be touched by them both, ſhall alſo be included in each of them.

Limitation. Then it's Diameter muſt not be given greater than the ſegment of the right line, joining the centers of the given circles, intercepted by their *concave* circumferences, which lies in the ſpace common to both the given circles; viz. not greater than CD.

There may be alſo three other Caſes of this Problem, when the given circles cut each other; but becauſe they are ſimilar to the 1ſt, 2d, and 4th Caſes already propoſed, and ſubject to juſt the ſame Limitations; except that which is ſimilar to the 1ſt, which is ſubject to no Limitation at all, they are here omitted; as are likewiſe thoſe Caſes where the given circles touch each other; becauſe they are the ſame as the preceding, and ſolved in the ſame manner.

The general Solution.

Join the given centers A and B, and where the Caſe requires, let AB be produced to meet the given circumferences in C and D: and let CI and DH be taken equal to the given line Z: and let two circles be deſcribed; one with center A and diſtance AI, and the other with center B and diſtance BH: and theſe two circles will neceſſarily cut or touch each other by the Limitations given. Let the point of concourſe be E: from E draw the right line EAF cutting the circle whoſe center is A in F; as alſo EBG cutting the circle whoſe center is B in G: then with center E and diſtance EF deſcribe a circle FK, this will be the circle required: becauſe AF and AC are equal as alſo AI and AE; therefore FE and CI are alſo equal: but CI was made equal to Z, therefore FE is equal to Z. Again, becauſe BD and BG are equal, as alſo BH and BE, therefore DH and EG are alſo equal; but DH was made equal to Z, therefore EG is equal to Z. Hence it appears that the circle FK, paſſing through F will alſo paſs thro' G, and likewiſe that it will alſo touch the given circles in F and G, becauſe EAF and EBG are right lines paſſing through the centers.

PROBLEM IV.

Having a given point A, and a given right line BC, it is required to draw a circle, whoſe Radius ſhall be equal to a given line Z, which ſhall paſs through the given point, and alſo touch the given line.

Limitation. 2Z muſt not be given leſs than the perpendicular let fall from the given point A upon the given line BC.

From the point A let AD be drawn perpendicular to BC, and in this perpendicular take DE equal to the given line Z: and through E draw EF parallel to BC, and from A upon this line EF ſet off AF equal to Z, which may be done, for by the Limitation Z is not leſs than AE: then with center F and diſtance FA deſcribe a circle, and I ſay it will touch the line BC: for through F drawing FG parallel to AD, FGDE will be a Parallelogram, and FG will be equal to DE, that is to Z, and at right angles to BC.

PROBLEM V.

Having a given point A, and alſo a given circle whoſe center is B, it is required to draw a circle whoſe Radius ſhall be equal to a given line Z, which ſhall paſs through the given point, and alſo touch the given circle.

This Problem has three Caſes, each of which is ſubject to a Limitation.

Case 1ſt. Let the circle to be deſcribed be required to be touched *outwardly* by the given circle.

Limitation. Then the Diameter muſt not be given leſs than the ſegment of the right line, joining the given point and the center of the given circle, which is intercepted between the given point and the *convex* circumference; viz. not leſs than AC.

Case 2d. Let the circle to be deſcribed be required to be touched *inwardly* by the given circle.

Limitation. Then the Diameter muſt not be given leſs than the right line which, drawn from the given point through the center of the given circle, is contained between the given point and the *concave* circumference; viz. not leſs than AC.

Case 3d. Let the given point lie in the given circle.

Limitation. Then a diameter of the given circle being drawn through the given point, it is divided into two ſegments by the ſaid point, and the Diameter of the circle required muſt not be given greater than the greater of them, nor leſs than the leſſer; viz. not greater than AC, nor leſs than AG.

The general Solution.

Let A and B be joined, and in the line AB take CF equal to Z, and then with center A and diſtance Z, let an arc be drawn, and with center B, and diſtance BF let another be drawn, which by the Limitations will neceſſarily either touch or cut the former; let the point of their concourſe be D; then with D center and DA diſtance let a circle be drawn, and I ſay it will touch the given circle whoſe center is B: for DB being drawn meeting the circumference of the circle whoſe center is B in E, BC is equal to BE, and hence CF equals ED, and they are both equal to the given line Z.

PROBLEM VI.

Having a right line given BC, and alſo a circle whoſe center is A, it is required to draw another circle, whoſe Radius ſhall be equal to a given right line Z, and which ſhall touch both the given line and alſo the given circle.

This Problem has alſo three caſes, each of which is ſubject to a Limitation.

Case 1ſt. Let the circle to be deſcribed be required to be touched *outwardly* by the given circle.

Limitation. Then the Diameter of the circle required muſt not be given leſs than the ſegment of a line, drawn from the center of the given circle, perpendicular to the given line, which is intercepted between the ſaid line and the *convex* circumference; viz. not leſs than BD.

Case 2d. Let the circle to be deſcribed be required to be touched *inwardly* by the given circle.

Limitation. Then the given line muſt not be in the given circle, neither muſt the Diameter of the circle required be given leſs than that portion of the perpendicular, drawn from the center of the given circle to the given line, which is intercepted between the ſaid line and the *concave* circumference; viz. not leſs than BD.

Case 3d. Let the circle to be deſcribed be required to be both touched and included in the given circle.

Limitation. Then the given right line muſt be in the given circle, and when a Diameter of this given circle is drawn cutting the given line at right angles, the Diameter of the circle required muſt not be given greater than the greater ſegment; viz. not greater than BD.

The general Solution.

From A draw AB perpendicular to BC, cutting the given circumference in D; and in this perpendicular let BG and DF be taken each equal to the given line Z; and through G draw GE parallel to BC; and with center A and diſtance AF let an arc be ſtruck, which by the Limitations will neceſſarily either touch or cut GE; let the point of concourſe be E, let AE be joined, and, if neceſſary, be produced to meet the given circumference in H; then with E center and EH diſtance deſcribe a circle, and I ſay it will be the required circle: it is evident it will touch the given circle: and becauſe AD and AH are equal, as alſo AF and AE, therefore DF (which was made equal to Z) will be equal to HE: let now EC be drawn perpendicular to BC, then GBCE will be a Parallelogram, and EC will be equal to GB, which was alſo made equal to Z: hence the circle will alſo touch the given line BC, becauſe the angle ECB is a right one.

PROBLEM VII.

Having two points A and B given in poſition, and likewiſe a right line EF given in poſition, it is required to find the center of a circle, which ſhall paſs through the given points and touch the given line.

Case 1ſt. When the points A and B are joined, ſuppoſe AB to be parallel to EF: then biſecting AB in D, and through D drawing DC perpendicular to it, DC will alſo be perpendicular to EF: draw a circle therefore which will paſs through the three points A, B, and C, (by Euc. IV. 5.) and it will be the circle required: (by a Corollary from Euc. III. 1. and another from Euc. III. 16.)

Case 2d. Suppoſe AB not parallel to EF, but being produced meets it in E: then from EF take the line EC ſuch, that its ſquare may be equal to the rectangle BEA, and through the points A, B, C, deſcribe a circle, and it will be the circle required by Euc. III. 37.

This is Vieta's Solution. But Mr. Thomas Simpſon having conſtructed this, and ſome of the following, both in the Collection of Problems at the end of his Algebra, and alſo among thoſe at the end of his Elements of Geometry, I ſhall add one of his Conſtructions.

Let A and B be the points given, and CD the given line: drawing AB and biſecting it in E, through E let EF be drawn perpendicular to AB and meeting CD in F: and from any point H in EF draw HG perpendicular to CD, and having drawn BF, to the ſame apply HI = HG, and parallel thereto draw BK meeting EF in K: then with center K and radius BK let a circle be deſcribed, and the thing is done: join KA, and draw KL perpendicular to CD, then becauſe of the parallel lines, HG : HI :: KL : KB; whence as HG and HI are equal, KL and KB are likewiſe equal. But it is evident from the Conſtruction that KA = KB, therefore KB = KL = KA.

Because two equal lines HI and Hi may be applied from H to BF each equal to HG, the Problem will therefore admit of two Solutions, as the Figure ſhews: except in the caſe when one of the given points, A for inſtance, is given in the line CD, for then the Problem becomes more ſimple, and admits but of one conſtruction, as the center of the circle required muſt be in the line EF continued, as alſo in the perpendicular raiſed from A to CD, and therefore in their common interſection: and this is the limit of poſſibility; for ſhould the line CD paſs between the given points, the Problem is impoſſible.

N. B. Tho' Vieta does not take notice that this Problem is capable of two anſwers, yet this is as evident from his conſtruction, as from Mr. Simpſon's, for EC (the mean proportional between EB and EA) may be ſet off upon the given line EF either way from the given point E.

LEMMA I.

A point A being given between the two right lines BC and DE, it is required through the point A to draw a line cutting the two given ones at equal angles.

If the given lines be parallel, then a perpendicular to them through the point A is the line required. But if not, then let them be produced to meet in the point F: and let FG be drawn biſecting the Angle BFD, and through A draw a perpendicular to FG, and it will be the line required by Euc. I. 26.

PROBLEM VIII.

Having a point A given, and alſo two right lines BC and DE, to draw a circle which ſhall paſs through the given point, and touch both the given right lines.

By the preceding Lemma draw a line IAH through the point A, which ſhall make equal angles with the two given lines BC and DE: biſect IH in K; and taking KL = KA, by means of the preceeding Problem draw a circle which ſhall paſs through the points A and L, and likewiſe touch one of the given lines, BC for inſtance, in the point M. I ſay this circle will alſo touch the other given line DE: for from the center N letting fall the perpendicular NO, and joining NI, NH, NM; in the triangles NKH, NKI, NK being common, and HK = KI, and the Angles at K right ones, by Euc. I. 4. NH = NI likewiſe the angle NHK = angle NIK, from hence it follows, that the angle NHM = angle NIO; and the angles at M and O, being both right, and NH being proved equal to NI, NM will be equal to NO by Euc. I. 26.

Mr. Simpſon conſtructs the Problem thus.

Let BD and BC be the given lines meeting in B, and A the given point, join AB, and draw BN biſecting the given angle DBC: and from any point E in BN upon BC let fall the perpendicular EF, and to BA apply EG = EF, parallel to which draw AH meeting BN in H: then from center H with Interval AH let a circle be deſcribed, and the thing is done. Upon BC and BD let fall the perpendiculars HI, HK, which are manifeſtly equal, becauſe by Conſtruction the angle HBI = HBK; moreover as EF : EG : : HI : HA : but EF and EG are equal, therefore alſo HI and HA.

PROBLEM IX.

Having a circle whoſe center is A given in magnitude and poſition, and alſo two right lines BD and ZC given in poſition, to draw a circle which ſhall touch all three.

From

FROM A draw two perpendiculars to the right lines DB, ZC; viz. ADF and AZX; and in theſe perpendiculars take DF, ZX, on either ſide of D and Z, equal to the Radius of the given circle: and through F and X draw lines parallel to DB, ZC; viz. FG, XH; and then by the preceding Problem draw a circle which ſhall paſs through the given center A and touch the two lines FG, XH; and E the center of this circle will alſo be the center of the circle required, as appears by ſubtracting equals from equals in Figure 1: and by adding equals to equals in Figure 2.

LEMMA II.

IF the two circles CEB and CED cut one another C, then I ſay a line drawn from the point of ſection CBD cutting both circles, will cut off diſſimilar ſegments from thoſe circles.

1ſt. SUPPOSE CB to be the Diameter of one of them: then draw to the other point of ſection E the line CE, and joining EB, ED, the angle CEB will be a right one, and the angle CED either greater or leſs than a right one, and conſequently CD cannot be a Diameter of the other.

2dly. SUPPOSE CBD not to paſs through the center of either: then through C draw a Diameter CAG, and continue it to meet the other circle in F, and join BG, DF: then the angle CBG is a right one, and the angle CDF is either greater or leſs than a right one:. and therefore the lines BG and DF are not parallel: let H be the center of the other circle, and let a Diameter CHI be drawn: draw DI and continue it meet to meet CG in K: then DIK will be parallel to BG: hence CB: CD:: CG: CK. But CI and CK are unequal, (being both applied from the ſame point in a right angle) and therefore it cannot be as CB: CD:: CG: CI: and hence it appears that the Segments CB and CD are diſſimilar.

LEMMA III.

IF through the legs of any triangle EDF (ſee Figure to Problem 10.) a line BI be drawn parallel to the baſe DF, ſo that there be conſtituted two ſimilar triangles about the ſame vertex; and a circle be circumſcribed about each of theſe triangles; theſe circles will touch one another in the common vertex E.

IT is plain that they will either touch or cut each other in the point E: if they cut each other, then by the preceding Lemma the Segments BE and DE would be diſſimilar; but they are ſimilar, and they muſt therefore touch each other.

PROBLEM X.

HAVING a point A, and alſo a right line BC, given in poſition; together with a circle whoſe center is G given both in magnitude and poſition; to deſcribe a circle a circle which ſhall paſs through the given point, and likewiſe touch both the given line and the given circle.

THE given right line may either touch or be entirely without the given circle, and then the given point muſt be ſo too; or it may cut the given circle, and then the given point muſt be within the ſame, or at leaſt in the circumference.

CASE 1ſt. Suppoſe the given line BC to touch or fall entirely without the given circle: through G the center of the given circle draw DGFC perpendicular to BC, and joining DA, take DH a 4th proportional to DA, DC, DF, ſo that DA × DH = DC × DF: then through the points A and H draw a circle touching the line CB by Problem VII, and I ſay it will alſo touch the given circle.

Draw DB cutting the given circle in E, and join FE. Now becauſe in the quadrilateral figure BCFE the oppoſite angles at C and E are each of them right ones, the figure will be in a circle, and therefore DC × DF = DB × DE. But DC × DF = DA × DH by Conſtruction. Hence DB × DE = DA × DH, and therefore the points B, E, H, A, will be alſo in a circle: but the point E is alſo in the given circle; therefore theſe circles either touch or cut one another in that point. Let now BI be drawn from the point of contact B perpendicular to the touching line BC to meet the circumference again in I, and it will be a Diameter: and let EI be joined: then becauſe the angles FED and BEI are vertical and each of them right ones, FEI will be a continued ſtraight line: and it appears that the two circles will touch each other by the preceding Lemmas.

CASE 2d. Suppoſe the given line BC to cut the given circle: the Conſtruction and Demonſtration are exactly the ſame as before, except that the angles FED and BEI are not vertical but coincident, and ſo EI is coincident with EF.

MR. SIMPSON has likewiſe ſolved this problem in a different manner, but I think his Conſtruction is neither ſo elegant nor eaſy as this of Vieta's; and it is therefore omitted.

PROBLEM XI.

HAVING two circles given in magnitude and poſition, whoſe centers are A and B, as likewiſe a right line CZ; to draw a circle which ſhall touch all three.

FROM the center of the leſſer circle B let BZ be drawn perpendicular to CZ, and in BZ (or in BZ continued as the caſe requires) let be taken ZX = AL the Radius of the other circle; and through X let XH be drawn parallel to CZ, and with center B and Radius BG, equal to the difference (or ſum as the Caſe requires) of the Radii of the two given circles, let a circle be deſcribed; and laſtly let another circle be deſcribed; touching this laſt and alſo the line XH and paſſing through the point A by Problem X, and I ſay that E the center of

this

this circle will alſo be the center of the circle required; as will appear by taking equals from equals, or adding equals to equals, as the aſſigned Caſe and Data ſhall require.

THE Caſes are four, though Vieta makes but three.

CASE 1ſt. If it be required that the circle ſhould touch both the others *externally* then BG muſt be taken equal to the *difference* of the Semidiameters of the two given circles, and ZX muſt be taken in BZ *produced.*

CASE 2d. If it be required that the circle ſhall touch and *include* both the given ones; then BG muſt be taken equal to the *difference*, as in Caſe 1ſt, but ZX muſt taken in BZ *itſelf.*

CASE 3d. If it be required that the circle ſhould touch *and include* the greater of the given circles, and touch *externally* the other whoſe center is B; then BG muſt be taken equal to the *ſum* of the Radii of the given circles, and ZX muſt be taken in BZ *itſelf.*

CASE 4th. If it be required that the circle ſhould touch the greater of the given circles *externally*, and touch and *include* the leſſer; then BG muſt be taken equal to the *ſum* of the Radii, and ZX muſt be taken in BZ *produced.*

PROBLEM XII.

HAVING two points given B and D, and likewiſe a circle whoſe center is A; to deſcribe another circle which ſhall paſs through the given points, and touch the given circle.

LET DB be joined, as alſo AB, and let AB be produced to cut the given circle in the points I and K, then let BH be taken a 4th proportional to DB, BK, BI; ſo that $BD \times BH = BI \times BK$: from H let a Tangent HF be drawn to the given circle; and BF be joined and cut the circle again in G: and let DG be drawn cutting the given circle again in E, and laſtly through the points D, B, G, let a circle be drawn, I ſay it will touch the given circle in G.

FOR joining EF; becauſe the Rectangle DBH = the Rectangle KBI, i. e. the Rectangle GBF, therefore the four points D, H, F, G, are in a circle; and hence the angle HFB = the angle GDB: (for in the two firſt figures one is the external angle of a quadrilateral figure, and the other is the internal and oppoſite, and in the two laſt figures theſe angles are in the ſame ſegment.) But the angle HFB = the angle GEF by Euc. III. 32. hence GEF = GDB: therefore the triangles GEF and GDB are ſimilar and under the ſame vertex, and therefore by Lemma 3. the circles deſcribed about them will touch each other in the common vertex G.

THERE are other Conſtructions of this Problem in Hugo de Omerique, Simpſon, and a periodical work ſtiled the Mathematician.

LEMMA IV.

Having two circles ABCI and EFGH given, it is required to find a point M, in the line joining their centers, or in that line continued, ſuch, that any line drawn through the ſaid point M, cutting both the circles, ſhall always cut off ſimilar ſegments.

Let the line KL joining the centers be ſo cut or produced, that KM may be to LM in the given ratio of the Radii AK to EL [in the caſe of producing KL we muſt make it as AK—EL : EL :: KL : LM, for then by compoſition it will be AK : EL :: KM : LM] and then I ſay that the point M will be the point required. For from it drawing any line MGFCB cutting the circle ABCI in B and C, and the circle EFGH in F and G; and let B and F be the correſpondent points moſt diſtant from M, and C and G the correſpondent points that are nearer; and let be joined BK, CK, FL, GL, and thereby two triangles will be formed BKC, FLG. Now becauſe by Conſtruction KM : LM :: KB : LF, KB and LF will be parallel by Euc. VI. 7. and for the ſame reaſon KC and LG will be parallel; and therefore the triangles BKC and FLG will be equiangled, and hence the ſegment BC will be ſimilar to the ſegment FG.

LEMMA V.

The point M being found as in the preceding Lemma, I ſay that it is a property of the ſaid point, that MG × MB = MH × MA : as alſo that MF × MC = ME × MI.

For joining CI and GH, it is evident that theſe lines will alſo be parallel. Hence MI : MC : : MH : MG, but MI : MC :: MB : MA, therefore MH : MG :: MB : MA, and MG × MB = MH × MA. Again MH : MG :: MF : ME, but MH : MG :: MI : MC, therefore MF : ME :: MI : MC and MF × MC = ME × MI.

PROBLEM XIII.

Having two circles given in magnitude and poſition, whoſe centers are K and L, and alſo a point D; to draw a circle which ſhall touch the two given ones, and alſo paſs through the point D.

Join the given centers by drawing KL, and in KL or KL produced find the point M (by Lemma 4.) ſuch, that all the lines drawn from it cutting the given circles ſhall cut off ſimilar ſegments; and let KL cut the circumferences, one of them in the points A and I, and the other in the points E and H; and joining MD, make it as MD : MA :: MH : MN. Then through the points D and N draw a circle which ſhall alſo touch the given circle whoſe center is K, by Prob. XII. and I ſay that this circle will alſo touch the other given circle whoſe center is L. For let B be the point of contact and BM be drawn cutting the circle K in C, and the circle L in F and G; then by Lemma 5. MB × MG =

MA × MH: but MA × MH = MD × MN by Conſtruction; therefore MB × MG = MD × MN, and the points B, G, N, D, are in a circle. But the point G is alſo in the circle L; therefore theſe circles either touch or cut each other in the point G. Now the circles BND and BCI touch one another in B by conſtruction; therefore the ſegment BC is ſimilar to the ſegment BG; and alſo by conſtruction the ſegment BC is ſimilar to the ſegment FG; and therefore the ſegment FG is ſimilar to the ſegment BG; and hence the circles FGE and BGD touch one another in the point G.

The Caſes are three.

Case 1ſt. If the circle be required to *touch and include* both the given ones; then M muſt be taken in KL *produced*; and MN muſt be taken a fourth proportional to MD, MA, MH, A being the moſt *diſtant* point of interſection in the circle K, and H the *neareſt* point of interſection in the circle L.

Case 2d. If the circle be required to touch both the given ones *externally*; then alſo M muſt be taken in KL *produced*; and MN taken a fourth proportional to MD, MA, MH, A being the *neareſt* point of interſection in the circle K, and H the moſt *diſtant* in the circle L.

Case 3d. If the circle be required to *touch and include* the circle K, and to touch L *externally*; then M muſt be taken in KL *itſelf*; and MN a fourth proportional to MD, MA, MH, A being the moſt *diſtant* point in K, and H the *neareſt* in L.

PROBLEM XIV.

Having three circles given whoſe centers are A, B, and D; to draw a fourth which ſhall touch all three.

Let that whoſe center is A be called the 1ſt, that whoſe center is B the 2d, and that whoſe center is D the 3d. Then with center B, and radius equal to the *difference*, or *ſum*, as the caſe requires, of the ſemidiameters of the 1ſt and 2d circles, let an auxiliary circle be deſcribed; and likewiſe with D center, and radius equal to the *difference*, or *ſum*, as the caſe requires, of the ſemidiameters of the 1ſt and 3d circles, let another auxiliary circle be deſcribed; and laſtly by the preceding Problem draw a circle which ſhall touch the two auxiliary ones, and likewiſe paſs through the point A which is the center of the firſt circle. Let the center of this laſt deſcribed circle be E, and the ſame point E will likewiſe be the center of the circle required; as will appear by adding equals to equals, or taking equals from equals, as the caſe requires.

The Caſes are theſe.

Case 1ſt. If it be required that the circle ſhould *touch and include* all the other three; then let A be the center of the greateſt given circle, B of the next, and D of the leaſt: and let BG = the *difference* of the ſemidiameters of the 1ſt and 2d, and DF =the *Difference* of the ſemidiameters of the 1ſt and 3d, and through A deſcribe a circle which ſhall touch the two auxiliary ones *externally*.

Case 2d. If it be required that the circle ſhould touch all the other three *externally*; then the circles being the ſame as before in reſpect to their magnitude, let BG and DF be alſo the ſame as in the 1ſt Caſe, but through A deſcribe a circle which ſhall *touch and include* the two auxiliary ones.

Case 3d. If it be required that the circle ſhould *touch and include* one of the given circles A, and touch the other two *externally*; then let BG and DF =the the *ſum* of the ſemidiameters of the 1ſt and 2d, and *ſum* of the ſemidiameters of of the 1ſt and 3d reſpectively; and through A deſcribe a circle which ſhall touch the two auxiliary ones *externally*.

Case 4th. If it be required that the circle ſhould touch *externally* one of the given circles A, and ſhould *touch and include* the other two; then let BG and DF= the *ſums* as in Caſe 3d, but through A deſcribe a circle which ſhall *touch and include* the two auxiliary ones.

SUP-

SUPPLEMENT.

PROBLEM I.

HAVING two points given A and B, to determine the Locus of the centers of all ſuch circles as may be drawn through A and B.

JOIN AB and biſect it in the point C, and through C, draw a perpendicular to it CE, and continue it both ways *in infinitum*, and it is evident that this line CE will be the Locus required.

PROBLEM II.

HAVING two right lines given AB and CD, to determine the Locus of the centers of all ſuch circles as may be drawn touching both the ſaid lines.

CASE 1ſt. Suppoſe AB and CD to be parallel; then drawing GI perpendicular to them both; biſect it in H, and through H draw EHF parallel to the two given lines, and it will be the Locus required.

CASE 2d. Suppoſe the given lines being produced meet each other in E, then biſecting the angle BED by the line EHF, this line EHF will be the Locus required.

PROBLEM III.

HAVING two circles given whoſe centers are A and B; to determine the Locus of the centers of all ſuch circles as ſhall touch the two given ones.

CASES 1ſt and 2d. Suppoſe it be required that the circles be touched *outwardly* by both the given ones; or that they be touched *inwardly* by both the given ones.

THEN joining the centers A and B, let AB cut the circumferences in C and D and produced in P and O: let CD which is intercepted between the *convex* circumferences be biſected in E: ſet off from E towards B the center of the greater

greater of the given circles the line EH = the difference of the Radii of the given circles, and with A and B as Foci and EH Tranverſe Axis, let two oppoſite Hyperbolas be deſcribed KEK and LHL: then I ſay that KEK will be the locus of the centers of all the circles which can be drawn ſo as to be touched *outwardly* by both the given circles; and LHL will be the locus of the centers of all the circles which can be drawn ſo as to be touched *inwardly* by both the given circles. For taking any point K in the Hyperbola KEK, and drawing KA and KB, let theſe lines cut the given circumferences in F and G reſpectively: and make KQ = KA: then from the nature of the curve QB = EH, but by conſtruction EH=BG—AF, therefore QB=BG—AF and hence QG = AF, and then taking equals from equals KG = KF, which is a demonſtration of the 1ſt Caſe.

THEN with regard to the 2d Caſe, taking any point L in the Hyperbola LHL, and drawing LB and LA and producing them to meet the *concave* circumferences in M and N, let alſo LR be taken equal to LB; then from the property of the curve AR = EH, but EH (by conſtruction) = BM—NA; therefore AR = BM—NA, and NR = BM, and then adding equals to equals LN = LM, which is a demonſtration of the 2d Caſe.

CASE 3d. Suppoſe it be required that the circles to be deſcribed be touched *outwardly* by one of the given circles, and *inwardly* by the other.

Then drawing AB, let it cut the *convex* circumferences in C and D, and the *concave* ones in P and O, biſect PD in E, and from E towards B ſet off EH = the ſum of the given radii. Then with A and B foci and EH tranſverſe axis, let two oppoſite hyperbolas be deſcribed KEK and LHL: and KEK will be the locus of the centers of the circles which are touched *inwardly* by the circle A and *outwardly* by the circle B; and LHL will be the locus of the centers of thoſe circles which are touched *inwardly* by B and *outwardly* by A. The demonſtration *mutatis mutandis* is the ſame as before.

CASE 4th. Suppoſe the given circle A to include B, and it be required that the circles to be deſcribed be touched *outwardly* by them both.

Let AB cut the circumferences in C and D, P and O: and biſecting CD in I, and ſetting off from I towards P, IL = the ſum of the ſemidiameters of the given circles, and with A and B foci, and IL tranſverſe axis, deſcribing an ellipſe LKI, it will be the locus of the centers of the circles required. For taking any point K in the ellipſe, and drawing AK and BK, let AK be continued to meet one of the given circumferences in G, and let BK meet the other F. Then from the property of the curve AK+BK=IL=AG+BF (by conſtruction::

ſtruction :) hence by ſubſtraction BK=KG+BF, and by ſubtraction again FK=KG.

Case 5th. Suppoſe the given circle A to include B, and it be required that the circles to be deſcribed be touched *outwardly* by A and *inwardly* by B.

Then let AB cut the circumferences in C and D, P and O: and biſecting CO in I, and ſetting off from I towards P, IL= the difference of the ſemidiameters of the given circles, and with A and B foci and IL tranſverſe axis deſcribing an ellipſe LKI, it will be the locus of the centers of the circles deſcribed, and the demonſtration, *mutatis mutandis*, is the ſame as in the laſt caſe.

Cases 6th and 7th. Suppoſe the two given circles cut each other, and it be required that the circles to be deſcribed either be touched and included in them both, or be touched by them both and at the ſame time include them both.

These two caſes are ſimilar to caſes 1ſt and 2d, and as there, ſo alſo here, the tranſverſe axis of the two oppoſite hyperbolas, which are the loci required, muſt be taken =the difference of the ſemidiameters of the given circles. The demonſtration is ſo alike, it need not be repeated.

PROBLEM IV.

Having a given point A, and a given right line BC, to determine the locus of the centers of thoſe circles which ſhall paſs through A and touch BC.

From A draw AG perpendicular to BC, then with focus A and directrix BC let a parabola be deſcribed, and it will be the locus required; for by the property of the curve FA always equals FG drawn perpendicular to the directrix.

PROBLEM V.

Having a given point A, and a given circle whoſe center is B, to determine the locus of the centers of all thoſe circles, which paſs through A, and at the ſame time are touched by the given circle.

Cases 1ſt and 2d. Suppoſe the point A to lie out of the given circle, and it be required that the circles to be deſcribed be either touched *outwardly* by the given circle, or *inwardly* by it.

Let AB be drawn, and let it cut the given circumference where it is *convex* towards A in the point C, and where it is *concave* in the point O: then biſecting AC in E, and ſetting off from E towards B, EH=BC the given radius, and with A and B foci and EH tranſverſe Axis deſcribing two oppoſite Hyperbolas KEK and LHL, it is evident that KEK will be the locus of the centers of thoſe circles which paſs thrugh A and are touched *outwardly* by the given circle, and LHL will be the locus of the centers of thoſe circles which paſs through A and are touched *inwardly* by the given circle.

Case 3d. Suppoſe the given point A to lie in the given circle, whoſe center is B.

Then joining AB and continuing it to meet the given circumference in C and O, biſect AC in E, and from E towards O ſetting off EH = BC the given Radius, and with A and B Foci and EH tranſverſe Axis deſcribing an Ellipſe EKH, it will evidently be the Locus required.

PROBLEM VI.

Having a right line BC given, and alſo a circle whoſe center is A, to determine the Locus of the centers of the circles which ſhall be touched both by the given right line and alſo by the given circle.

There are three Caſes, but they are all comprehended under one general ſolution.

Case 1ſt. Let the given right line be without the given circle, and let it be required that the circles to be deſcribed be touched *outwardly* by the given circle.

Case 2d. Let the given right line be without the given circle, and let it be required that the circles to be deſcribed, be touched *inwardly* by the given circle.

Case 3d. Let the given right line be within the given circle, and then the circles to be deſcribed muſt be touched *outwardly* by the given circle.

General Solution.

From the given center A let fall a perpendicular AG to the given line BC, which meets the given circumference in D [or in Caſes 2d and 3d is produced to meet it in D] and biſecting DG in F, and ſetting off FM = FA (which is the ſame thing as making GM = AD the given Radius) and through M drawing MLK parallel to the given line BC, with A Focus and LK Directrix deſcribe a Parabola, and it will be the Locus of the centers of the circles required; for from the property of the Curve FA = FM, and adding equals to equals, or ſubtracting equals from equals, as the Caſe requires, FD = FG.

FINIS.